Calorons in the Weyl gauge

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We demonstrate by explicit construction that while the untwisted Harrington-Shepard caloron A_μ is manifestly periodic in Euclidean time, with a period $\beta=1/T$, when transformed to the Weyl ($A_0=0$) gauge, the caloron gauge field A_i is periodic only up to a large gauge transformation, with the winding number equal to the caloron's topological charge. This helps clarify the tunneling interpretation of these solutions, and their relation to Chern-Simons numbers and winding numbers.

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Instantons have many important applications in particle physics [1–6]. They may be defined as classical solutions to the Euclidean Yang-Mills equations that minimize the Yang-Mills action within a given topological charge sector. Explicit instanton solutions [7] can be found by solving the first-order self-duality equation

$$F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{1}$$

At zero temperature, instantons provide a semiclassical description of tunneling between classical minima of the Yang-Mills potential that are connected by large gauge transformations [8,9]. At finite temperature, the simplest instanton solutions are the untwisted ''calorons'' of Harrington and Shepard [10]. These calorons are solutions to the self-duality equation (1) that are periodic in Euclidean time, with a period $\beta=1/T$ equal to the inverse temperature. At first sight, it may seem puzzling that a periodic solution, with $A_\mu(\vec{x},t=-\beta/2)=A_\mu(\vec{x},t=+\beta/2)$, could describe tunneling from one sector to another since the gauge field A_μ is the same at either end. The answer is very simple but illustrative, and does not appear to have been spelled out in detail before. In the Weyl ($A_0=0$) gauge, which is most convenient for describing Hamiltonian processes such as tunneling, the coordinate gauge field A_i is *not* periodic; rather, $A_i(\vec{x},t=+\beta/2)$ is related to $A_i(\vec{x},t=-\beta/2)$ by a fixed-time large gauge transformation whose winding number is equal to the integer topological charge of the caloron solution. Here we present a simple explicit construction of this gauge transformation, yielding the Weyl gauge form of the caloron. We conclude with some comments concerning the distinction between the caloron solutions and the so-called ''periodic instantons'' studied in [11–14]. For simplicity, we concentrate in this paper on untwisted calorons [10], which are related to tunneling between sectors of different Chern-Simons charge; we will not address twisted calorons [15–17], which involve tunneling between different flux sectors. Twisted calorons are important for Yang-Mills-Higgs theory with symmetry breaking, and especially for understanding the monopole content of the theory. The twisted calorons have nontrivial holonomy, characterized by a nontrivial Polyakov loop at spatial infinity: $\mathcal{P}\exp(\int_0^\beta dt A_0)$, and the expectation value of A_0 at infinity plays the role of a Higgs expectation value [15–17]. Thus, it is simpler to discuss first the Weyl gauge for untwisted calorons which have a trivial holonomy.

In discussing SU(2) instantons, it is convenient to use a radially symmetric ansatz form [18]. Here, we adopt the conventions of [19] for the symmetric ansatz for the SU(2) instanton field $A_\mu=(\sigma^a/2i)A_\mu^a$:

$$A_i^a=\frac{\phi_2-1}{r^2}\epsilon_{iak}x_k+\frac{\phi_1}{r}\left(\delta_{ia}-\frac{x_ix_a}{r^2}\right)+A_1\frac{x_ix_a}{r^2}$$
$$A_0^a=A_0\frac{x_a}{r} \tag{2}$$

where A_0, A_1, ϕ_1, and ϕ_2 are functions of $r=\sqrt{\vec{x}^2}$ and t only. The charge 1 instanton for SU(2) can then be expressed in terms of a single function $\rho(r,t)$, with the ansatz functions in Eq. (2) given by

$$A_0=\partial_r \ln\rho$$
$$A_1=-\partial_0 \ln\rho$$
$$\phi_1=-r\,\partial_0 \ln\rho$$
$$\phi_2=1+r\partial_r \ln\rho. \tag{3}$$

Then the gauge field described by Eqs. (2),(3) satisfies the instanton equation (1) if ρ satisfies the linear equation: $\Box\rho=0$. For the zero temperature, charge 1, instanton in SU(2), the function $\rho(r,t)$ can be chosen:

$$\rho=1+\frac{\lambda^2}{r^2+t^2}. \tag{4}$$

This form corresponds to a single instanton located at the origin in R^4, and with scale parameter λ. This choice is known as the ''singular gauge.''

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To facilitate the transformation of the gauge field in Eq. (2) into the Weyl ($A_0=0$) gauge, we use the fact [18] that the ansatz fields A_0, A_1, ϕ_1, and ϕ_2, which appear in Eqs. (2) and (3), transform in an Abelian-like manner under a particular type of SU(2) gauge transformation on the non-Abelian gauge field $A_\mu=(\sigma^a/2i)A_\mu^a$. Namely, if the gauge transformation matrix U has the special form

$$U(\vec{x},t)=e^{-(i/2)f(r,t)\hat{x}\cdot\vec{\sigma}}=\cos\left(\frac{f}{2}\right)-i(\hat{x}\cdot\vec{\sigma})\sin\left(\frac{f}{2}\right) \qquad (5)$$

then the SU(2) gauge transformation, $A_\mu\rightarrow\tilde{A}_\mu=U^{-1}A_\mu U+U^{-1}\partial_\mu U$, has the following effect on the ansatz functions A_0, A_1, ϕ_1, and ϕ_2:

$$A_0\rightarrow\tilde{A}_0=A_0+\partial_0 f$$

$$A_1\rightarrow\tilde{A}_1=A_1+\partial_r f$$

$$\phi_1\rightarrow\tilde{\phi}_1=\cos f\,\phi_1+\sin f\,\phi_2$$

$$\phi_2\rightarrow\tilde{\phi}_2=-\sin f\,\phi_1+\cos f\,\phi_2. \qquad (6)$$

Thus, to achieve the Weyl gauge, $\tilde{A}_0^a=0$, we simply need a gauge transformation $U(\vec{x},t)$ of the form in Eq. (5) where the function $f(r,t)$ satisfies

$$\partial_0 f=-\partial_r \ln\rho. \qquad (7)$$

For the zero temperature, charge 1, instanton with ρ given by Eq. (4), this leads to

$$f(r,t)=\left(\pi+2\arctan\frac{t}{r}\right)-\frac{r}{\sqrt{r^2+\lambda^2}}\left(\pi+2\arctan\frac{t}{\sqrt{r^2+\lambda^2}}\right) \qquad (8)$$

where we have chosen the constant of integration so that $f(r,t=-\infty)=0$. The arctan function is understood to take values in the $[-\pi/2,\pi/2]$ branch. The gauge field, $\tilde{A}_i^a$, in Weyl gauge has the ansatz form (2), with ansatz functions $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\tilde{A}_1$, (and $\tilde{A}_0\equiv 0$) given by Eq. (6).

The tunneling interpretation of the instanton solution becomes evident by noting that $\tilde{A}_i$ is pure gauge at $t=\pm\infty$. At $t=-\infty$,

$$\tilde{A}_i^a(\vec{x},t=-\infty)=0 \qquad (9)$$

while at $t=+\infty$,

$$\begin{aligned}\tilde{A}_i^a(\vec{x},t=+\infty)&=\frac{\cos\left(\frac{2\pi r}{\sqrt{r^2+\lambda^2}}\right)-1}{r^2}\epsilon_{iak}x_k\\&\quad-\frac{\sin\left(\frac{2\pi r}{\sqrt{r^2+\lambda^2}}\right)}{r}\left(\delta_{ia}-\frac{x_i x_a}{r^2}\right)\\&\quad-\frac{2\pi\lambda^2}{(r^2+\lambda^2)^{3/2}}\frac{x_i x_a}{r^2}\\&=(W^{-1}\partial_i W)^a\end{aligned} \qquad (10)$$

where the static gauge function $W(\vec{x})$ is

$$W(\vec{x})=e^{-i/2f(r,t=+\infty)\hat{x}\cdot\vec{\sigma}}=-\exp\left[i\frac{\pi r}{\sqrt{r^2+\lambda^2}}\hat{x}\cdot\vec{\sigma}\right]. \qquad (11)$$

Thus, as t goes from $t=-\infty$ to $t=+\infty$, the instanton gauge field $\tilde{A}_i$ in Weyl gauge interpolates between the pure gauge $\tilde{A}_i=0$ in Eq. (9), and the pure gauge $\tilde{A}_i=W^{-1}\partial_i W$ in Eq. (10). These fields $\tilde{A}_i$ are neighboring minima (in fact, zeros) of the classical Yang-Mills potential

$$V=\frac{1}{2}\int d^3x\, B_i^a B_i^a. \qquad (12)$$

From Eqs. (9) and (10) we see that these two minima are related by the large gauge transformation $W(\vec{x})$ in Eq. (11). Furthermore, the fixed-time gauge transformation in Eq. (11) is $W(\vec{x})=U(\vec{x},t=+\infty)$, where $U(\vec{x},t)$ is the (time-dependent) gauge transformation used to transform to the Weyl gauge. To evaluate the winding number of the fixed-time gauge transformation W, we note that if a gauge transformation has the form $W(\vec{x})=e^{-(i/2)g(r)\hat{x}\cdot\vec{\sigma}}$, then the winding number N is determined [21] by the values of $g(r)$ at $r=0$ and $r=\infty$:

$$\begin{aligned}N&\equiv\frac{1}{24\pi^2}\int d^3x\,\epsilon_{ijk}\,\mathrm{tr}(W^{-1}\partial_i W W^{-1}\partial_j W W^{-1}\partial_k W)\\&=-\frac{1}{2\pi}[g(r)-\sin g(r)]_{r=0}^{r=\infty}.\end{aligned} \qquad (13)$$

Thus, for the gauge transformation in Eq. (11), we find $N=1$.

Having reviewed this familiar tunneling interpretation of zero temperature instantons, we now turn to finite temperature instantons, or ''calorons.'' At finite T, the Euclidean time direction is compactified onto the finite interval $t\in[-\beta/2,+\beta/2]$, where $\beta=1/T$ is the inverse temperature. The manifold is now $R^3\times S^1$, rather than R^4. Instantons are solutions to the same self-duality equation (1), but the gauge fields, being bosonic, satisfy periodic boundary conditions [20,3] on the Euclidean time interval $[-\beta/2,+\beta/2]$.

The untwisted Harrington-Shepard (HS) caloron solution is consistent with the ansatz form described in Eqs. (2) and (3). The only change is that the function $\rho(r,t)$ becomes [10]

$$\rho(r,t)=1+\frac{1}{2}\left(\frac{2\pi\lambda}{\beta}\right)^2\left(\frac{\beta}{2\pi r}\right)\frac{\sinh\left(\frac{2\pi r}{\beta}\right)}{\cosh\left(\frac{2\pi r}{\beta}\right)-\cos\left(\frac{2\pi t}{\beta}\right)}$$

$$\equiv 1+\frac{1}{2}\frac{\bar{\lambda}^2}{\bar{r}}\frac{\sinh\bar{r}}{\cosh\bar{r}-\cos\bar{t}} \tag{14}$$

where we have defined the convenient shorthand: $\bar{r}=2\pi r/\beta$, etc. Note that ρ in Eq. (14) satisfies $\Box\rho=0$. The caloron solution given by Eqs. (2), (3) and (14) corresponds to an infinite line of zero temperature instantons, each of scale λ and each at spatial position $\vec{x}=0$, lined up along the t axis, with period β. It is straightforward to check that in the zero temperature limit ($\beta\to\infty$), the expression for $\rho(r,t)$ in Eq. (14) reduces smoothly to that in Eq. (4), and we recover the zero temperature instanton. At finite T, the caloron solution (2),(3) is manifestly periodic in t, $A_\mu(\vec{x},t+\beta)=A_\mu(\vec{x},t)$, since $\rho(r,t)$ in Eq. (14) has period β.

To explore the tunneling interpretation of this caloron solution, we transform it to the Weyl gauge. Once again, the Weyl gauge can be attained most easily by using the abelian-like transformation properties (6) of the ansatz functions in Eq. (2). The required SU(2) transformation matrix has the form in Eq. (5), with $f(r,t)$ satisfying Eq. (7), and ρ is now given by Eq. (14). The solution to Eq. (7) in this case is

$$f(r,t)=\left(\pi+2\arctan\left[\coth\left(\frac{\bar{r}}{2}\right)\tan\left(\frac{\bar{t}}{2}\right)\right]\right)+\frac{(\bar{\lambda}^2-2\bar{r}^2)\sinh\bar{r}-\bar{\lambda}^2\bar{r}\cosh\bar{r}}{\bar{r}\sqrt{\sinh\bar{r}(4\bar{\lambda}^2\bar{r}\cosh\bar{r}+(\bar{\lambda}^4+4\bar{r}^2)\sinh\bar{r})}}$$

$$\times\left(\pi+2\arctan\left[\tan\left(\frac{\bar{t}}{2}\right)\frac{\bar{\lambda}^2\sinh\bar{r}+2\bar{r}(1+\cosh\bar{r})}{\sqrt{\sinh\bar{r}(4\bar{\lambda}^2\bar{r}\cosh\bar{r}+(\bar{\lambda}^4+4\bar{r}^2)\sinh\bar{r})}}\right]\right) \tag{15}$$

where the constant of integration has been chosen so that $f(r,t=-\beta/2)=0$. Note that this function $f(r,t)$, and hence the gauge transformation $U(\vec{x},t)=\exp[-(i/2)f(r,t)\hat{x}\cdot\vec{\sigma}]$ that transforms the HS caloron to the Weyl gauge, is manifestly periodic in t, with period β. It is straightforward to check that $f(r,t)$ in Eq. (15) reduces smoothly, as $\beta\to\infty$, to the zero temperature function in Eq. (8).

Given the function $f(r,t)$ in Eq. (15), the caloron fields $\tilde{A}_i^a$ in the Weyl gauge are obtained simply from Eqs. (2) and (6). To investigate the relation of this caloron solution to quantum tunneling, we evaluate $\tilde{A}_i^a$ at $t=\pm\beta/2$. At $t=-\beta/2$, the ansatz fields satisfy $\tilde{A}_1=\tilde{\phi}_1=0$, while $\tilde{\phi}_2=[1+r\partial_r\ln\rho(r,t=-\beta/2)]$, where $\rho(r,t=-\beta/2)=1+(\bar{\lambda}^2/2\bar{r})\tanh(\bar{r}/2)$. Thus,

$$\tilde{A}_i^a\left(\vec{x},t=-\frac{\beta}{2}\right)=-\frac{\lambda^2\pi\left(\tanh\left(\frac{\pi r}{\beta}\right)-\frac{\pi r}{\beta}\operatorname{sech}^2\left(\frac{\pi r}{\beta}\right)\right)}{\beta r^3\left(1+\frac{\lambda^2\pi}{r\beta}\tanh\left(\frac{\pi r}{\beta}\right)\right)}\epsilon_{iak}x_k. \tag{16}$$

For low temperatures,

$$\tilde{A}_i^a\left(\vec{x},t=-\frac{\beta}{2}\right)\sim-\frac{2}{3}\lambda^2\pi^4T^4\epsilon_{iak}x_k \tag{17}$$

which is nonzero, but which vanishes at zero temperature, as in Eq. (9). At $t=+\beta/2$, the expression for $\tilde{A}_i^a(\vec{x},t=+\beta/2)$ is more cumbersome to write out, but it is completely specified by the ansatz form (2) with

$$\tilde{\phi}_1\left(r,t=+\frac{\beta}{2}\right)=\sin f\left(r,t=+\frac{\beta}{2}\right)\phi_2\left(r,t=+\frac{\beta}{2}\right)$$

$$\tilde{\phi}_2\left(r,t=+\frac{\beta}{2}\right)=\cos f\left(r,t=+\frac{\beta}{2}\right)\phi_2\left(r,t=+\frac{\beta}{2}\right)$$

$$\tilde{A}_1\left(r,t=+\frac{\beta}{2}\right)=\partial_r f\left(r,t=+\frac{\beta}{2}\right) \tag{18}$$

and where f is given by Eq. (15), and ϕ_2 is as in Eq. (3).

We now ask how $\tilde{A}_i^a(\vec{x},t=+\beta/2)$ is related to $\tilde{A}_i^a(\vec{x},t=-\beta/2)$. Since the ansatz functions satisfy $\tilde{A}_1(r,t=-\beta/2)=0$, and $\phi_2(r,t=+\beta/2)=\tilde{\phi}_2(r,t=-\beta/2)$, this shows that $\tilde{A}_i^a(\vec{x},t=+\beta/2)$ and $\tilde{A}_i^a(\vec{x},t=-\beta/2)$ are related by a gauge transformation

$$\tilde{A}_i\left(\vec{x},t=+\frac{\beta}{2}\right)=W^{-1}\tilde{A}_i\left(\vec{x},t=-\frac{\beta}{2}\right)W+W^{-1}\partial_i W \tag{19}$$

where the static large gauge transformation $W(\vec{x})$ is

$$W(\vec{x})=e^{-(i/2)f(r,t=+\beta/2)\hat{x}\cdot\vec{\sigma}}=-\exp\left[i\pi\frac{(\bar{\lambda}^2-2\bar{r}^2)\sinh\bar{r}-\bar{\lambda}^2\bar{r}\cosh\bar{r}}{\bar{r}\sqrt{\sinh\bar{r}(4\bar{\lambda}^2\bar{r}\cosh\bar{r}+(\bar{\lambda}^4+4\bar{r}^2)\sinh\bar{r})}}(\hat{x}\cdot\vec{\sigma})\right]. \tag{20}$$

Using Eq. (13), the winding number of this gauge transformation is

$$N=-\frac{1}{2\pi}\left[f\left(r,t=+\frac{\beta}{2}\right)-\sin f\left(r,t=+\frac{\beta}{2}\right)\right]_{r=0}^{r=\infty}=1 \tag{21}$$

for *all* inverse temperatures β and scales λ.

Notice that Eq. (19) shows that in the Weyl gauge the gauge fields $\tilde{A}_i$ are *not* strictly periodic. Rather, $\tilde{A}_i$ is periodic up to a large gauge transformation. This fixed-time gauge transformation is $W(\vec{x})=U(\vec{x},t=+\beta/2)$, where $U(\vec{x},t)$ is the time-dependent gauge transformation used to transform to the Weyl gauge. So, just as in the $T=0$ case, the initial and final gauge fields, $\tilde{A}_i(\vec{x},t=\mp\beta/2)$, are related by a static large gauge transformation of unit winding number. But at finite T, these initial and final gauge fields $\tilde{A}_i(\vec{x},t=\mp\beta/2)$ are not pure gauge. This means that the tunneling at finite T is not between *minima* of the classical Yang-Mills potential (12), but between fields of higher potential energy. With the ansatz (2), the corresponding magnetic field strength is (and since the instanton solution is self-dual, this also gives the electric field strength)

$$B_i^a=-\frac{(\partial_r\phi_1-A_1\phi_2)}{r^2}\epsilon_{iak}x_k+\frac{(\partial_r\phi_2+A_1\phi_1)}{r}\left(\delta_{ia}-\frac{x_ix_a}{r^2}\right)+\frac{(1-\phi_1^2-\phi_2^2)}{r^4}x_ix_a. \tag{22}$$

It is straightforward to verify that $\tilde{B}_i^a(\vec{x},t)$ is nonvanishing at $t=\mp\beta/2$. The magnetic potential energy (12) can be expressed in terms of the ansatz functions in Eqs. (2) and (3) [since V is gauge invariant, it doesn't matter whether we use the original gauge or the Weyl gauge]:

$$V=2\pi\int_0^\infty dr\left\{2(\partial_r\phi_1-A_1\phi_2)^2+2(\partial_r\phi_2+A_1\phi_1)^2+\frac{1}{r^2}(1-\phi_1^2-\phi_2^2)^2\right\}. \tag{23}$$

At zero temperature, $V(t=\mp\infty)=0$, since the zero temperature instanton is a pure gauge at $t=\mp\infty$. But at finite temperature, $V(t=\mp\beta/2)$ is nonzero, even though it approaches zero as $\beta\to\infty$. It is straightforward to evaluate numerically V in Eq. (23) for the zero and finite temperature instantons discussed above. In Fig. 1 we plot the potential energy as a

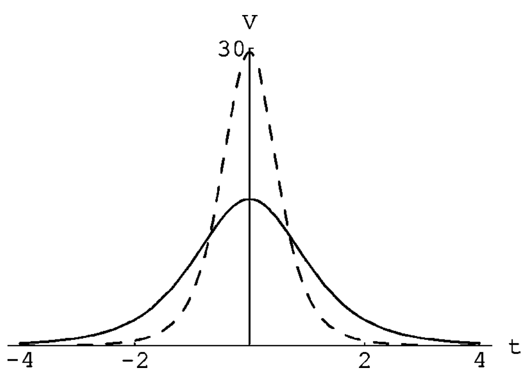

FIG. 1. The potential energy (23) as a function of t, at zero temperature. The dashed line is for instanton scale parameter $\lambda=1$, while the solid line has $\lambda=2$. Note that $V\to 0$ at $t=\pm\infty$, as the Weyl gauge solutions are pure gauge at $t=\pm\infty$.

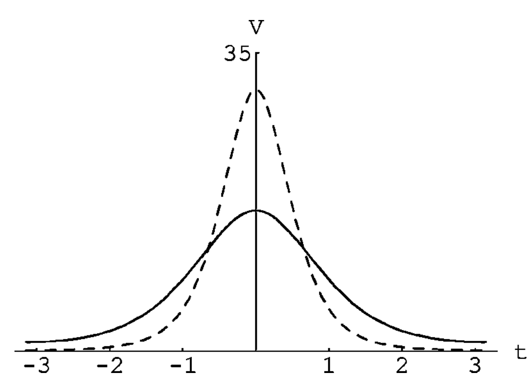

FIG. 2. The potential energy (23) as a function of $\bar{t}=2\pi t/\beta$, at finite temperature. The dashed line is for scale parameter $\bar{\lambda}=1$, while the solid line is for $\bar{\lambda}=2$. Note that V does not vanish at $\bar{t}=\pm\pi$, as the Weyl gauge solutions are not pure gauge at $t=\pm\beta/2$.

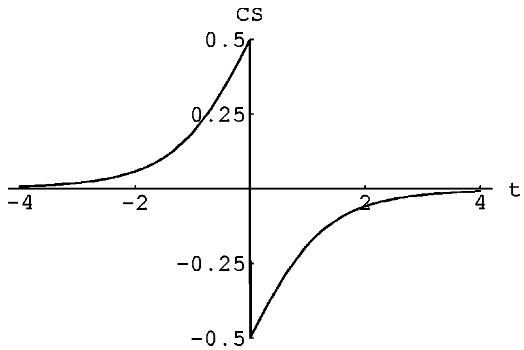

FIG. 3. Plot of the Chern-Simons function $CS(t)$ in Eq. (26), in the singular gauge at zero temperature, as a function of t. The instanton scale factor λ has been scaled to 1.

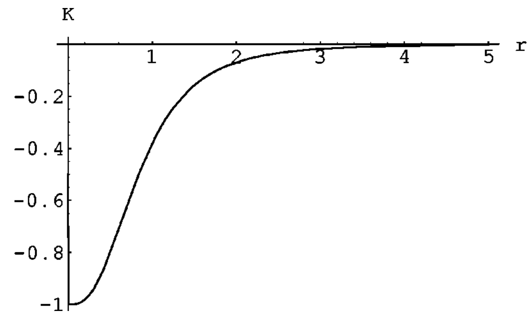

FIG. 4. Plot of the function $K(r)$ in Eq. (27), in the singular gauge at zero temperature, as a function of r. The instanton scale factor λ has been scaled to 1.

function of Euclidean time for the zero temperature instanton for which $\rho(r,t)$ is given by Eq. (4), and in Fig. 2 for the finite temperature caloron for which $\rho(r,t)$ is given by Eq. (14).

We now consider the instanton's topological charge and its relation with Chern-Simons numbers and winding numbers. Recall that the instanton charge may be expressed as

$$Q=-\frac{1}{32\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$
$$\equiv\int d^4x\ \partial_\mu K_\mu ,$$

where

$$K_\mu=-\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left(A_\nu\partial_\rho A_\sigma+\frac{2}{3}A_\nu A_\rho A_\sigma\right). \tag{24}$$

This leads to a natural separation of Q as

$$Q=\int dt\frac{d}{dt}CS(t)+\int_0^\infty dr\frac{d}{dr}K(r). \tag{25}$$

In terms of the ansatz fields in Eq. (2), the Chern-Simons term, $CS\equiv\int d^3xK_0$, is

$$CS(t)=\frac{1}{2\pi}\int_0^\infty dr[\phi_1\phi_2'-\phi_2\phi_1'-A_1(1-\phi_1^2-\phi_2^2)+\phi_1'] \tag{26}$$

while

$$K(r)=\frac{1}{2\pi}\int dt[-\phi_1\dot{\phi}_2+\phi_2\dot{\phi}_1+A_0(1-\phi_1^2-\phi_2^2)-\dot{\phi}_1]. \tag{27}$$

Here ϕ' and $\dot{\phi}$ denote derivative with respect to r and t respectively. At zero temperature, the t integration in Eq. (27) is from $-\infty$ to $+\infty$, while at finite temperature it is over $[-\beta/2,+\beta/2]$. Note that both $CS(t)$ and $K(r)$ are gauge dependent, while Q itself is gauge invariant. Also note that the last terms in each of $CS(t)$ and $K(r)$ cancel one another in the decomposition (25), but are required for the correct transformation properties of the respective terms. For example, under a gauge transformation of the form (5), the Chern-Simons term changes as

$$\widetilde{CS}=CS+\frac{1}{8\pi^2}\int d^3x\ \epsilon_{ijk}\partial_i\,\mathrm{tr}(\partial_jUU^{-1}A_k)$$
$$+\frac{1}{24\pi^2}\int d^3x\ \epsilon_{ijk}\,\mathrm{tr}(U^{-1}\partial_iUU^{-1}\partial_jUU^{-1}\partial_kU)$$
$$=CS+\frac{1}{2\pi}\int_0^\infty dr\frac{d}{dr}[(\phi_2-1)\sin f+\phi_1(\cos f-1)]$$
$$-\frac{1}{2\pi}\int_0^\infty dr\frac{d}{dr}[f-\sin f]. \tag{28}$$

It is a simple but instructive exercise to check that this non-Abelian transformation law agrees with the transformation of Eq. (26) under the Abelian-like transformations (6) of the ansatz functions, provided one includes the ϕ_1' term in $CS(t)$ [22].

For the instanton solutions studied in this paper, one can evaluate numerically the functions $CS(t)$ and $K(r)$ in Eqs. (26) and (27), in the original gauge (3), and in the Weyl gauge $\tilde{A}_i$ where $\tilde{A}_0=0$. At zero temperature, with the scalar function $\rho(r,t)$ given by Eq. (4), the functions $CS(t)$ and $K(r)$ are plotted in Figs. 3 and 4. In this gauge, $CS(t=$

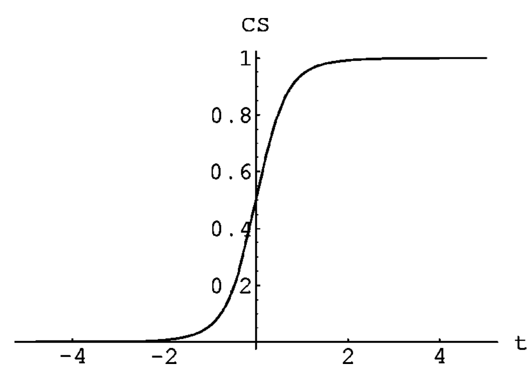

FIG. 5. Plot of the Chern-Simons function $CS(t)$ in Eq. (26), in the Weyl gauge at zero temperature, as a function of t. The instanton scale factor λ has been scaled to 1.

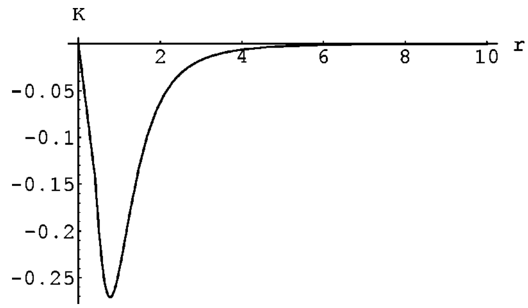

FIG. 6. Plot of the function $K(r)$ in Eq. (27), in the Weyl gauge at zero temperature, as a function of r. The instanton scale factor λ has been scaled to 1.

$+\infty)-CS(t=-\infty)=0$, and $K(r=\infty)-K(r=0)=1$. Thus, in this gauge the entire contribution to the instanton charge Q comes from the $K(r)$ term in the decomposition (25). On the other hand, in the Weyl gauge, the functions $\widetilde{CS}(t)$ and $\widetilde{K}(r)$ are plotted in Figs. 5 and 6. In the Weyl gauge, the entire contribution to Q comes from the change in the Chern-Simons number. Also note that in the Weyl gauge the Chern-Simons number is an integer at $t=\pm\infty$; this is because at $t=\pm\infty$ the Weyl gauge field is pure gauge [see Eqs. (9) and (10)], and for a pure gauge field the Chern-Simons term is equal to the winding number of the corresponding group element. The time-dependent gauge transformation $U=\exp(-(i/2)f(r,t)\hat{x}\cdot\vec{\sigma})$, with $f(r,t)$ in Eq. (8), that transforms from the original singular gauge to the Weyl gauge can be considered to have a t-dependent winding number $N(t)$ that, from Eqs. (13) and (8), is a step function in time:

$$N(t)=\frac{1}{2}(1+\mathrm{sgn}(t)). \tag{29}$$

This is consistent with the plots in Figs. 3 and 5, since the middle term in Eq. (28) vanishes in this case. Thus, in the Weyl gauge, the Chern-Simons number provides a meaningful label for the distinct classical wells of the Yang-Mills potential. At $t=0$, the Weyl gauge Chern-Simons number is $\frac{1}{2}$, corresponding to the sphaleron at the peak of the barrier between neighboring wells. The winding number of the gauge transformation $U(\vec{x},t)$ is also $\frac{1}{2}$ when $t=0$.

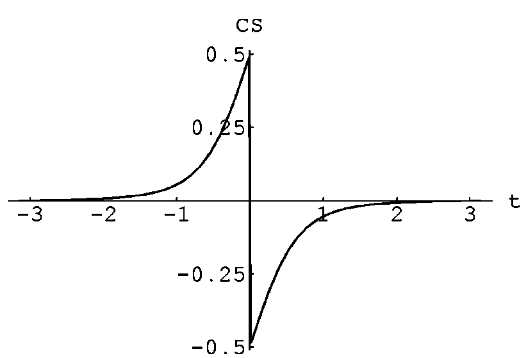

FIG. 7. Plot of the Chern-Simons function $CS(t)$ in Eq. (26), in the Harrington-Shepard gauge at finite temperature, as a function of $\bar{t}=2\pi t/\beta$. The scale factor has been chosen $\bar{\lambda}=1$.

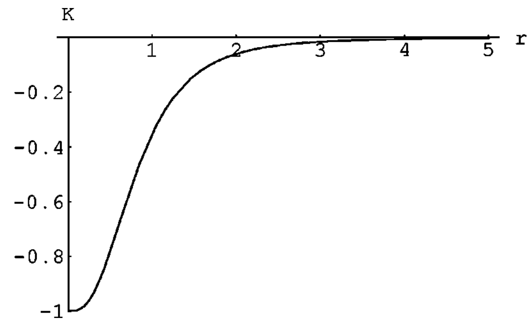

FIG. 8. Plot of the function $K(r)$ in Eq. (27), in the Harrington-Shepard gauge at finite temperature, as a function of $\bar{r}=2\pi r/\beta$. The scale factor has been chosen $\bar{\lambda}=1$.

At finite temperature, the situation is similar. The only change is that the scalar function $\rho(r,t)$ is now given by Eq. (14), and the t integration in Eq. (27) is over the finite interval $[-\beta/2,+\beta/2]$. In the Harrington-Shepard gauge (3),(14), the functions $CS(t)$ and $K(r)$ are plotted in Figs. 7 and 8. (These plots, and those in Figs. 9 and 10, are for scale parameter $\bar{\lambda}=1$. For different $\bar{\lambda}$, the plots have the same form.) Once again, in this gauge, the entire contribution to the instanton charge comes from the change in $K(r)$. The corresponding plots for the Weyl gauge are in Figs. 9 and 10, and we see that the entire contribution to Q comes from the change $CS(t)$. From the fact (19) that in the Weyl gauge $\widetilde{A}_i(t=+\beta/2)$ is related to $\widetilde{A}_i(t=-\beta/2)$ by a static gauge transformation of winding number 1, we know that the difference $\widetilde{CS}(t=+\beta/2)-\widetilde{CS}(t=-\beta/2)=1$. But it is interesting to note further that in the Weyl gauge the Chern-Simons number is an integer at $t=\pm\beta/2$, even though the gauge field is *not* a pure gauge [see Eqs. (16) and (19)]. At $t=0$, the Weyl gauge Chern-Simons number is $\frac{1}{2}$, as is the winding number of the gauge transformation $U(\vec{x},t)$ that connects the HS gauge to the Weyl gauge.

In conclusion, we have demonstrated explicitly that while the HS caloron solution is manifestly periodic, so that $A_\mu(\vec{x},t=+\beta/2)=A_\mu(\vec{x},t=-\beta/2)$, when transformed to the Weyl ($A_0=0$) gauge, the gauge field is periodic up to a large gauge transformation, $\widetilde{A}_i(\vec{x},t=+\beta/2)=W^{-1}\widetilde{A}_i(\vec{x},t=$

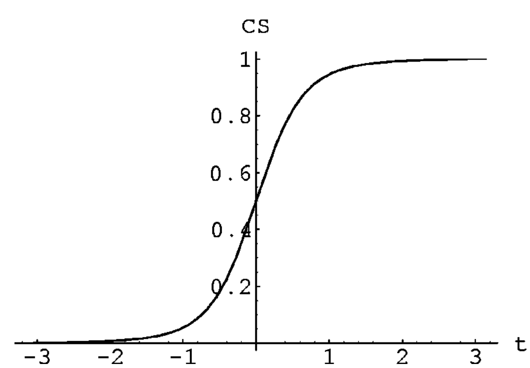

FIG. 9. Plot of the Chern-Simons function $CS(t)$ in Eq. (26), in the Weyl gauge at finite temperature, as a function of $\bar{t}=2\pi t/\beta$. The scale factor has been chosen $\bar{\lambda}=1$.

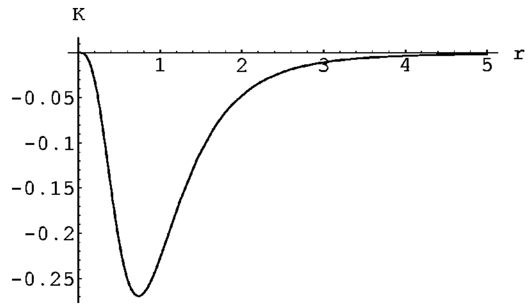

FIG. 10. Plot of the function $K(r)$ in Eq. (27), in the Weyl gauge at finite temperature, as a function of $\bar{r}=2\pi r/\beta$. The scale factor has been chosen $\bar{\lambda}=1$.

$-\beta/2)W+W^{-1}\partial_i W$, where W is a fixed-time gauge transformation with integer winding number equal to the caloron's topological charge. This gauge transformation W connects different Chern-Simons-number sectors of the classical Yang-Mills potential energy, just as is familiar from zero temperature. However, the caloron solution in Weyl gauge interpolates, as t goes from $-\beta/2$ to $+\beta/2$, not between classical minima but between classical gauge fields of *non-zero* potential energy.

Having understood the tunneling interpretation of the HS calorons in the Weyl gauge, we conclude with some brief comments concerning the difference between these calorons and the ''periodic instantons'' studied in [11–14]. These are very different objects, but the names can be somewhat confusing since calorons are themselves instantons that are periodic in Euclidean time. This distinction, and some of its physical consequences, have been stated clearly in [11], but we reiterate these distinctions here in the light of our Weyl gauge calorons. The ''periodic instantons'' are most easily introduced in the simple context of a quantum mechanical model; for example, a double well potential $V=(q^2-1)^2/4$. This captures most of the essential physics. This potential model has well-known instanton solutions [23] that are periodic in Euclidean time t:

$$q(t)=\sqrt{\frac{2\nu}{1+\nu}}\,\mathrm{sn}\left(\frac{t}{\sqrt{1+\nu}}-K\,;\nu\right), \tag{30}$$

where $\mathrm{sn}(x;\nu)$ is a Jacobi elliptic function [24], and the modulus parameter $0\leqslant\nu\leqslant 1$ is related to the energy by $E=(\nu-1)^2/(4(\nu+1)^2)$. These solutions are manifestly periodic: $q(t=-\beta/2)=q(t=+\beta/2)$, where the period $\beta=4\sqrt{1+\nu}\,K(\nu)$, and $K(\nu)$ is the elliptic quarter-period [24]. Physically, these periodic instantons correspond to tunneling from one well to the next, and then back again to the original starting position (since they are periodic). By contrast, the Weyl gauge calorons describe tunneling from one classical well (characterized by Chern-Simons number 0) to the next well (with CS=1); they do not tunnel back again, since the Weyl gauge field $\tilde{A}_i(t=+\beta/2)$ differs from $\tilde{A}_i(t=-\beta/2)$ by a large gauge transformation, as we have demonstrated explicitly in Eqs. (19),(20). This difference between calorons and periodic instantons also manifests itself through the fact that the calorons have nonzero topological charge per period (the explicit ones studied here have Q=1), while the ''periodic instantons'' have zero net topological charge per period. For low temperatures, the periodic instantons can be approximated by an infinite chain of alternating zero temperature instantons and anti-instantons. Indeed, this is particularly easy to see in the quantum mechanical double-well potential case (30), owing to the remarkable elliptic function identity:

$$\mathrm{sn}\left(\frac{t}{\sqrt{1+\nu}};\nu\right)=\frac{\pi}{2\sqrt{\nu}K'}\sum_{n=-\infty}^{\infty}(-1)^n \times\tanh\left(\frac{\pi}{2\sqrt{1+\nu}K'}\left(t-n\frac{\beta}{2}\right)\right). \tag{31}$$

Here $K'(\nu)\equiv K(1-\nu)$. This leads to the simple interpretation of the periodic instanton in Eq. (30) as a series of alternating instantons and anti-instantons. The zero temperature limit corresponds to $\nu\to 1$, in which case the period $\beta\to\infty$, and $E\to 0$, and we approach the usual zero temperature kink-like instantons. But, on the full period $[-\beta/2,\beta/2]$, the sn function reduces to a *pair* of an anti-kink and a kink. This explains why the net topological charge per period is zero. It is only on the half-period $[0,\beta/2]$ that $q(t)$ approaches the familiar zero temperature instanton: $q=\tanh(t/\sqrt{2})$. By contrast, the HS caloron is a periodic sum of instantons (and an HS anti-caloron is a periodic sum of anti-instantons), *not* a periodic sum of alternating instantons and anti-instantons. Indeed, in Yang-Mills theory there is no known exact ''periodic instanton'' that is an exact alternating chain of instantons and anti-instantons, although approximate chains of this form have been extensively studied [11,5]. This difference between calorons and periodic instantons can be reconciled because even though the caloron solutions are manifestly periodic, $A_\mu(t+\beta)=A_\mu(t)$, since this is a gauge theory, the A_μ are not all independent physical fields; when we transform the caloron to the Weyl gauge we see that the coordinate fields $\tilde{A}_i$ are periodic only up to a large gauge tranformation. We hope that our explicit construction of the Weyl gauge caloron fields helps to clarify this important distinction between calorons and ''periodic instantons.''

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